Rutgers University: Algebra Written Qualifying Exam

January 2011: Day 1 Problem 6 Solution

Exercise. Prove there are no simple groups of order 80.

Solution.

Let G be a group of order 80. We want to show that there is a normal subgroup of G that is ***not*** $\{e\}$ or G. So first find the prime factors of $|G| = 80$.

$$80 = 2^4 \cdot 5.$$

By the third Sylow theorem,

$$\begin{array}{llll} n_2 \equiv 1 \mod 2 & \text{and} & n_2 \mid 5 & \implies & n_2 = 1 \text{ or } 5 \\ n_5 \equiv 1 \mod 5 & \text{and} & n_5 \mid 16 & \implies & n_5 = 1 \text{ or } 16 \end{array}$$

If the number of $5-$Sylow subgroups is $n_5 = 1$, then the $5-$Sylow subgroup is a normal subgroup of G by the Second Sylow Theorem.
Thus, G is not simple.

If $n_5 \neq 1$ then there are $n_5 = 16$ $5-$Sylow subgroups.
$\implies$ G has $16(5-1) = 64$ elements of order 5.
Therefore, G has $80 - 64 = 16$ other elements.
These must be the elements of the 2-Sylow subgroup, which has order $2^4 = 16$.
Thus, the number of $2-$Sylow subgroup must be $n_2 = 1$.
$\implies$ the 2-Sylow subgroup is a normal subgroup of G by the Second Sylow Theorem.
Thus, G is not simple.